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East coast ocean currents

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A three-dimensional barotropic model of the wind-driven ocean circulation is examined and the flow near the east coast of the ocean is considered in detail. The model is linear and has constant coefficients of eddy viscosity. It is shown that a strong current may exist in the eastern boundary layer when upwelling or downwelling is present. No net northwards transport is produced as an equal deep counter current also occurs. A consequence of the downwelling is that the principal interior gyre is prevented from reaching the east coast and a secondary gyre is formed in this region.

1. Introduction

Recent work on the wind-driven ocean circulation using three-dimensional models has shown that two-dimensional transport theories, although adequate to describe many features of the circulation, actually mask a number of interesting effects. The model considered here is a rectangular ocean with uniform depth on a β plane with vertical lateral boundaries. The co-ordinate system used has x increasing eastwards, y northwards and z vertically upwards, with corresponding velocity components u, v, w and f is the variable Coriolis parameter. A wind stress (τ^x, τ^y) is applied to the surface at $z = 0$. In such a homogeneous ocean, there are two mechanisms that may drive the interior flow between the Ekman layer. If curl $(\boldsymbol{\tau}/f)$ is non-zero, a horizontal divergence is produced in the surface Ekman layer causing Ekman layer suction (upwelling from the interior) which in turn drives the interior horizontal motion. This process determines v and w but leaves some arbitrariness in u. The second mechanism occurs if the wind stress has a northward component. This induces an eastward Ekman transport which descends in the eastern boundary layer and enters the interior as a further contribution to the u velocity (the weak lower Ekman layer being unable to accept this volume of fluid).

If the wind stress has the form $\tau^x = -\tau \cos y$, $\tau^y = 0$ there is no eastward Ekman transport and no downwelling at the east coast. The interior flow is determined entirely by curl $(\boldsymbol{\tau}/f)$. Examination of the western boundary layer shows the usual westward intensification as in transport theories. However, when the boundary layer is matched to the interior solution, Johnson (1968) found that some water in the northward flowing west coast current in the sub-tropical gyre leaves this gyre and flows north-eastwards to enter the subpolar gyre. Upwelling occurs in the northern gyre and downwelling in the southern gyre. These effects are masked by transport theories for which only closed gyres are found.

On the other hand if the wind stress is $\tau^x = 0$, $\tau^y = \tau(y)$ then curl $(\boldsymbol{\tau}/f)$ is zero and v and w are zero in the interior. However, as shown by Pedlosky (1968), an eastward Ekman transport is produced which descends in the eastern boundary layer, enters the interior where it flows westwards, and ascends in the western boundary layer. Transport theories merely predict no net transport.

In this paper we consider a combination of the above two effects by applying a wind stress $\boldsymbol{\tau} = \boldsymbol{\tau}(x, y)$. In addition to transport between gyres and upwelling (or downwelling) at the coasts, some additional effects are found that are absent in transport theories. Associated with upwelling near the east coast of the ocean, there may be an east coast current which, although weak compared with the west coast current, is significant compared with the interior flow. A countercurrent is produced at lower depths. As these results depend on the magnitude of a constant horizontal eddy viscosity, a rather controversial concept, and as no account has been taken of density stratification, it is unrealistic to expect this model to represent too closely the actual ocean circulation. However, it is interesting to note here that a similar kind of circulation is observed off the coasts of California, south-west Africa (the Benguela current) and Peru–Chile when upwelling occurs. A secondary effect due to the upwelling and independent of the magnitude of the eddy viscosity is the production of an extra gyre near the east coast.

A much more realistic model would include a variable density distribution. However, the numerical work of Bryan & Cox (1967) shows that the effect of the wind stress is confined to a thermal boundary layer just below the Ekman layer, and moreover the distribution of upwelling and horizontal velocities have many features in common with the circulations discussed here. More recently Pedlosky (1969) in a linear stratified theory shows that an important contribution to the circulation in the thermal boundary layer is induced by the surface wind stress and that the lateral boundary layers are important in determining the interior flow. This suggests that it is profitable to examine in detail the mathematically much simpler homogeneous model to predict features which may then be looked for in more complicated and more realistic models.

2. Equations of motion

The linear momentum and continuity equations for steady flow in dimensionless variables are

$$-fv = -P_x + E_H(u_{xx} + u_{yy}) + E_V u_{zz}, \tag{1}$$

$$fu = -P_y + E_H(v_{xx} + v_{yy}) + E_V v_{zz}, \tag{2}$$

$$0 = -P_z + \delta^2 E_H(w_{xx} + w_{yy}) + \delta^2 E_V w_{zz}, \tag{3}$$

$$u_x + v_y + w_z = 0, \tag{4}$$

where $f = 1 + \beta y$ is the variable Coriolis parameter used in the β-plane approximation, P is the reduced pressure and u, v, w are the velocity components eastwards, northwards and vertically upwards respectively. The small parameters are defined as

$$E_H = \nu_H/f_0 L^2, \quad E_V = \nu_V/f_0 D^2, \quad \delta = D/L,$$

where D, L are vertical and horizontal length scales, and ν_V, ν_H are constant vertical and horizontal eddy viscosities. A more detailed derivation of these equations is given in Johnson (1968). As the magnitude of the eddy viscosities, and hence the Ekman numbers E_H, E_V, are difficult to determine, we shall write

$$E_H = E_V = E.$$

No qualitative change in the nature of the solutions occurs by keeping separate Ekman numbers.

The boundary conditions at the surface are

$$u_z = E^{-\frac{1}{2}}\tau^x, \quad v_z = E^{-\frac{1}{2}}\tau^y, \quad w = 0; \quad z = 0.$$

At the bottom and on the east and west coasts, the velocity is zero so that

$$u = v = w = 0; \quad z = -1, \quad x = 0, 1.$$

The north and south boundaries remain free, but we shall confine our attention to a region between latitudes of zero wind stress curl.

3. The Ekman layer and interior solutions

The solution in the upper Ekman layer away from the coasts is standard and is given by

$$u_E = \frac{1}{(2f)^{\frac{1}{2}}}\exp\{-\zeta(\tfrac{1}{2}f)^{\frac{1}{2}}\}\{(\tau^y-\tau^x)\sin\zeta(\tfrac{1}{2}f)^{\frac{1}{2}}+(\tau^y+\tau^x)\cos\zeta(\tfrac{1}{2}f)^{\frac{1}{2}}\}+O(E^{\frac{1}{2}}), \quad (5)$$

$$v_E = \frac{1}{(2f)^{\frac{1}{2}}}\exp\{-\zeta(\tfrac{1}{2}f)^{\frac{1}{2}}\}\{(\tau^y-\tau^x)\cos\zeta(\tfrac{1}{2}f)^{\frac{1}{2}}-(\tau^y+\tau^x)\sin\zeta(\tfrac{1}{2}f)^{\frac{1}{2}}\}+O(E^{\frac{1}{2}}), \quad (6)$$

$$w_E(x,y,\infty) = E^{\frac{1}{2}}\mathbf{k}.\mathrm{curl}\,(\boldsymbol{\tau}/f), \quad (7)$$

where $\mathbf{k} = (0,0,1)$ and $z = -E^{\frac{1}{2}}\zeta$. Each solution in this paper will be valid in only the region indicated and will be matched to solutions in neighbouring regions. Consequently small differences may be noted compared with corresponding solutions in Pedlosky (1968) where correction functions are used.

The lowest-order solution in the interior region, away from the coasts and below the Ekman layer, is

$$u_I = \frac{E^{\frac{1}{2}}}{f}\frac{\partial}{\partial y}\int_x^1 \frac{f^2}{\beta}\mathbf{k}.\mathrm{curl}\left(\frac{\boldsymbol{\tau}}{f}\right)dx + E^{\frac{1}{2}}U(y), \quad (8)$$

$$v_I = E^{\frac{1}{2}}\frac{f}{\beta}\mathbf{k}.\mathrm{curl}\left(\frac{\boldsymbol{\tau}}{f}\right), \quad (9)$$

$$w_I = E^{\frac{1}{2}}(z+1)\,\mathbf{k}.\mathrm{curl}\left(\frac{\boldsymbol{\tau}}{f}\right), \quad (10)$$

where $U(y)$ has to be found from matching with the eastern boundary layer.

The lower Ekman layer is weak and serves only to smooth the horizontal velocity components to zero.

4. The east coast boundary layer

To obtain the structure of the east coast boundary layer, we assume that $\partial/\partial x \gg \partial/\partial y$, $\partial/\partial z$ and simplify equations (1)–(3) to

$$\begin{aligned} -fv &= -P_x + E_H u_{xx}, \\ fu &= -P_y + E_H v_{xx}, \\ 0 &= -P_z + \delta^2 E_H w_{xx}. \end{aligned}$$

Elimination of the variables in favour of v leads to

$$\delta^2 E_H^2 v_{xxxxxx} + E_H^2 v_{xxxxzz} - \delta^2 E_H \beta v_{xxx} + f^2 v_{zz} = 0. \tag{11}$$

The solution of this equation has been considered by Pedlosky (1968) for $E_H \gg \delta^2$. However, the magnitude of E_H is difficult to assess owing to the choice of a reasonable value for ν_H and it is likely that δ and $E_H^{\frac{1}{2}}$ will have similar orders of magnitude. Consequently it is useful to examine the opposite limit $E_H \ll \delta^2$ as a number of distinctive results are found.

In the remainder of this section and in §§ 5 and 6 it will be assumed that $E_H \ll \delta^2$. Under this restriction the second term of (11) is always smaller than the other terms. The first and third terms of (11) balance in a boundary layer of thickness $E_H^{\frac{1}{3}}$ provided $v_z \equiv 0$, which requires that the solution is hydrostatic. The first and fourth terms of (11) balance in a boundary layer of thickness $(\delta E_H)^{\frac{1}{3}}$ a non-hydrostatic layer in which upwelling occurs.

The outer $E^{\frac{1}{3}}$ layer

In this hydrostatic layer the appropriate stretched co-ordinate is $\eta = (1-x)\,E^{-\frac{1}{3}}$ and the variables may be expanded as follows,

$$\begin{aligned} u &= E^{\frac{1}{2}}\hat{u} + \ldots, \quad & v &= E^{\frac{1}{6}}\hat{v} + O(E^{\frac{1}{2}}), \\ w &= E^{\frac{1}{2}}\hat{w} + \ldots, \quad & P &= E^{\frac{1}{2}}\hat{P} + \ldots. \end{aligned}$$

Substitution into (1)–(4) gives

$$f\hat{v} = -\hat{P}_\eta, \quad f\hat{u} = -\hat{P}_y + \hat{v}_{\eta\eta}, \quad \hat{P}_z = 0, \quad \hat{u}_\eta = \hat{v}_y. \tag{12}$$

Elimination of $\hat{P}$, $\hat{u}$ gives the equation for $\hat{v}$ as

$$\hat{v}_{\eta\eta\eta} + \beta\hat{v} = 0, \tag{13}$$

which may be compared with the first and third terms of (11). The solution of this equation which is bounded as $\eta \to \infty$ is

$$\hat{v} = \hat{v}_1(y)\, e^{-\beta^{\frac{1}{3}}\eta} \tag{14}$$

and the continuity equation gives

$$\hat{u} = \hat{u}_1(y) - \beta^{-\frac{1}{3}}\hat{v}_1'(y)\, e^{-\beta^{\frac{1}{3}}\eta},$$

where the prime denotes differentiation. Matching with the interior solution (8) gives

$$\hat{u}_1(y) = E^{-\frac{1}{2}} u_I(1, y) = U(y). \tag{15}$$

As v is smaller than $O(E^{\frac{1}{6}})$ in the inner layer, matching (14) with the inner layer gives $\hat{v}_1 = 0$. Hence the $E^{\frac{1}{3}}$ layer on the east coast does not exist to the lowest order. It is of course important on the west coast where it provides the north–south return flow.

The inner $(\delta E)^{\frac{1}{3}}$ layer

For this layer we introduce the stretched co-ordinate $\xi = (1-x)(\delta E)^{-\frac{1}{3}}$ and the following expansions for the other variables:

$$\left.\begin{aligned} u &= E^{\frac{1}{2}}\bar{u}_0 + \delta E^{\frac{1}{2}}\bar{u}_1 + \ldots, \\ v &= \delta^{\frac{2}{3}}E^{\frac{1}{6}}\bar{v}_0 + \ldots, \end{aligned}\right\} \tag{16}$$

$$\left.\begin{aligned} w &= (E^{\frac{1}{6}}/\delta^{\frac{1}{3}})\bar{w}_0 + \delta^{\frac{2}{3}}E^{\frac{1}{6}}\bar{w}_1 + \ldots, \\ P &= E^{\frac{1}{2}}\bar{P}_0 + \delta E^{\frac{1}{2}}\bar{P}_1 + \ldots. \end{aligned}\right\} \tag{17}$$

The choice of expansion has been made to bring in the non-hydrostatic nature of the layer and so that u and P can match with the outer solution. The magnitude of w is chosen so that this layer can accept the Ekman transport from the surface Ekman layer. The substitution of these expansions into (1)–(4) leads to

$$\bar{P}_{0\xi} = 0, \quad -f\bar{v}_0 = \bar{P}_{1\xi}, \tag{18}$$

$$f\bar{u}_0 = -\bar{P}_{0y} + \bar{v}_{0\xi\xi}, \tag{19}$$

$$\bar{P}_{0z} = 0, \quad \bar{P}_{1z} = \bar{w}_{0\xi\xi}, \tag{20}$$

$$\bar{u}_{0\xi} - \bar{w}_{0z} = 0. \tag{21}$$

We see that $\bar{P}_0$ depends only on y and is the hydrostatic part of the pressure field which matches with the outer flow, whereas $\bar{P}_1$ is the non-hydrostatic contribution. The inclusion of the extra term in (20) compared with (12) shows that lateral friction is more important in this layer than in the $E^{\frac{1}{3}}$ layer. Elimination among (18)–(21) leads to a single equation for this layer,

$$\left(\frac{\partial^6}{\partial \xi^6} + f^2\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial \bar{u}_0}{\partial z}, \bar{v}_0, \bar{w}_0, \bar{P}_1\right) = 0, \tag{22}$$

which may be compared with the first and fourth terms of (11).

The above series expansions also give the following information. As $E \ll \delta^2$ the largest velocity component is w, the upwelling or downwelling. As $E^{\frac{1}{6}} \gg \delta^{\frac{2}{3}}E^{\frac{1}{6}} \gg E^{\frac{1}{2}}$ the northward velocity component v is larger than the interior flow but less than the $O(E^{\frac{1}{6}})$ northwards flow in the west coast current. Thus if $\bar{v}_0 \neq 0$ there will be an appreciable current along the east coast. Finally, as $(\delta E)^{\frac{1}{3}} \gg E^{\frac{1}{2}}$ the boundary conditions on w are the Ekman layer compatibility conditions

$$\bar{w}_0 = 0 \quad \text{on} \quad z = 0, -1; \quad \xi \neq 0. \tag{23}$$

The boundary conditions at the east coast are

$$\bar{u}_0 = \bar{v}_0 = \bar{w}_0 = 0; \quad \xi = 0. \tag{24}$$

The matching conditions for large ξ are, using (15),

$$\bar{u}_0 \to U(y), \quad \partial\bar{u}_0/\partial z \to 0, \quad \bar{v}_0 \to 0, \quad \bar{w}_0 \to 0 \quad \text{as} \quad \xi \to \infty. \tag{25}$$

Another restriction comes from the fact that the Ekman transport into the corner region near $z = 0$, $x = 1$ must descend in this eastern boundary layer and then enter the interior as no volume flux of this order can be accepted by the lower Ekman layer. That is

$$\int_{\infty}^{0} -u_E(1,y,\zeta)\,E^{\frac{1}{2}}d\zeta = \int_{\infty}^{0} E^{\frac{1}{2}}\overline{w}_0(\xi,y,0)\,d\xi = \int_{-1}^{0} -u_I(1,y)\,dz$$

or
$$\frac{\tau^y(1,y)}{f} = \int_{0}^{\infty} -\overline{w}_0(\xi,y,0)\,d\xi = -U(y). \qquad (26)$$

The solution of (22) subject to the conditions (23)–(26) is found by separation of variables and to simplify the presentation the details are given in an appendix. The complete solution in the inner $(\delta E)^{\frac{1}{3}}$ layer is

$$\overline{u}_0 = -[\tau^y(1,y)/f]\left\{1+\sum_{n=1}^{\infty}[4/\sqrt{3}]\cos(n\pi z)\exp(-\tfrac{1}{2}\gamma_n\xi)\sin(\tfrac{1}{2}\sqrt{3}\,\gamma_n\xi+\tfrac{1}{3}\pi)\right\}, \qquad (27)$$

$$\overline{v}_0 = \tau^y(1,y)\sum_{n=1}^{\infty}[4/\sqrt{3}]\,\gamma_n^{-2}\cos(n\pi z)\exp(-\tfrac{1}{2}\gamma_n\xi)\sin\tfrac{1}{2}\sqrt{3}\,\gamma_n\xi, \qquad (28)$$

$$\overline{w}_0 = \tau^y(1,y)\sum_{n=1}^{\infty}[4/\sqrt{3}]\,\gamma_n^{-2}\sin(n\pi z)\exp(-\tfrac{1}{2}\gamma_n\xi)\sin\tfrac{1}{2}\sqrt{3}\,\gamma_n\xi, \qquad (29)$$

where $\gamma_n = (n\pi f)^{\frac{1}{3}}$. These series are uniformly convergent except near $z = 0$, $\xi = 0$ where the downwelling fluid leaves the Ekman layer. The same kind of singularity occurs in Stewartson layers. To examine the singularity in detail would require an analysis of the $E^{\frac{1}{2}} \times E^{\frac{1}{2}}$ corner region.

The velocity component $\overline{w}$ is large and provides the upwelling and downwelling near the coast. The vertical transport in this boundary layer at a given value of z is

$$\int_{0}^{\infty} E^{\frac{1}{2}}\overline{w}_0\,d\xi = \frac{2E^{\frac{1}{2}}\tau^y(1,y)}{f}\sum_{n=1}^{\infty}\frac{\sin n\pi z}{n\pi}$$
$$= -\frac{E^{\frac{1}{2}}\tau^y(1,y)}{f}(1+z) \quad (-1 \leqslant z < 0),$$

from Tolstov (1962, p. 102). This shows that the vertical transport decreases linearly with depth so that none enters the lower Ekman layer. Upwelling occurs when $\tau^y(1,y)$ is negative and there is a southward component of wind stress near the east coast.

The velocity component $\overline{u}$ serves to bring the interior flow to zero at the boundary and conveys the downwelling water into the interior. The boundary condition $\overline{u}_0 = 0$ at $\xi = 0$ is satisfied as

$$\sum_{n=1}^{\infty}\cos n\pi z = -\tfrac{1}{2} \quad (-1 \leqslant z < 0),$$

from Tolstov (1962, p. 170).

The velocity component $\overline{v}$ is large compared with the interior velocities and from the form of the series (16) and (17) we see that this current arises as a secondary effect of the upwelling. The horizontal northward transport at a given depth z in this layer is proportional to

$$\int_0^\infty \bar{v}_0 \, d\xi = \frac{2\tau^y(1,y)}{f} \sum_{n=1}^{\infty} \frac{\cos n\pi z}{n\pi}$$

$$= -\frac{2\tau^y(1,y)}{\pi f} \log|2\sin\tfrac{1}{2}\pi z| \quad (z \neq 0, \pm 2, \ldots),$$

from Tolstov (1962, p. 91). This profile is shown in figure 1. *If there is a southward component of wind stress near the coast the east coast current is southwards near the surface and northwards at greater depth. This is associated with upwelling.*

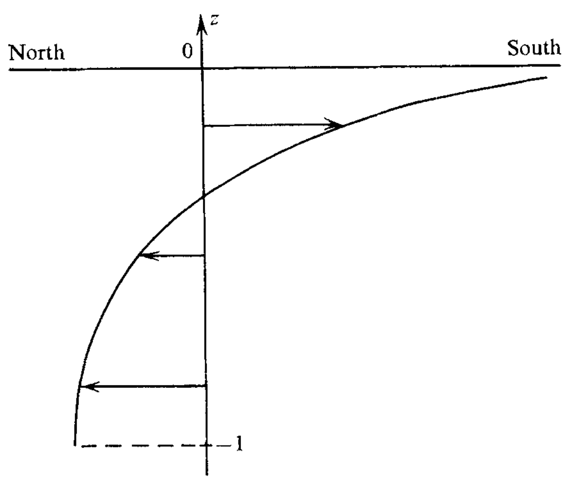

FIGURE 1. Variation with depth of northward transport in the east coast current when there is a southward component of wind stress.

The directions are reversed when the wind stress has a northward component. The singularity at $z = 0$ is not important as $\log|2\sin\frac{1}{2}\pi z|$ is $O(1)$ when $|z|$ is $O(E^{\frac{1}{2}})$ for $E = O(10^{-6})$. This requires an $O(1)$ contribution to v in the corner region where $|z| < O(E^{\frac{1}{2}})$. The total northward transport in this layer is

$$\int_{-1}^{0}\int_0^\infty \bar{v}_0 \, d\xi \, dz = -\frac{2\tau^y(1,y)}{\pi f}\int_{-1}^{0} \log|2\sin\tfrac{1}{2}\pi z| \, dz = 0,$$

which shows that the flow in the east coast current near the surface is just balanced by the counter-current at greater depth.

5. Example of an east coast current

Let the local wind stress near the east coast ($x = 1$) be southwards and have the form

$$\tau^x = 0, \quad \tau^y = -\cos k(x-1).$$

The interior flow outside the eastern boundary layer from (8)–(10) and (26) is

$$u_I = (E^{\frac{1}{2}}/f)\cos k(x-1), \quad v_I = (E^{\frac{1}{2}}/\beta)\, k \sin k(x-1),$$

$$w_I = (E^{\frac{1}{2}}/f)\, k(z+1)\sin k(x-1).$$

This represents a flow that is towards the coast. Further away from the coast the flow is southwards with associated slow downwelling.

In the east coast boundary layer there is upwelling, a southwards current near the surface and a northwards current nearer the bottom. As the southwards component of the interior tends to zero as the coast is approached this situation shows up the coastal current particularly clearly. Figure 2 shows these results diagrammatically.

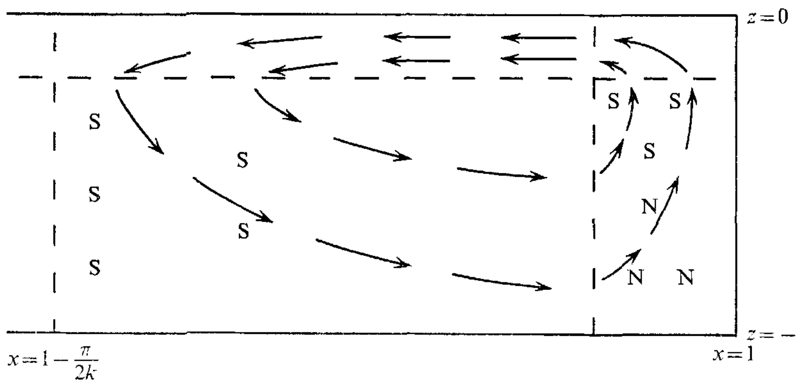

FIGURE 2. Diagrammatic representation of upwelling and zonal flow (shown by arrows) near the east coast at $x = 1$ produced by the wind stress $\tau^x = 0, \tau^y = -\cos k(x-1)$. The northward and southward flow is indicated by the letters N and S respectively.

Whilst stressing that these results depend on the condition that $E_H \ll \delta^2$ and that density variations have been omitted from this model, it is still interesting to compare the currents represented in figure 2 with observations in the ocean. The current directions derived here correspond with those found in the California current when upwelling is present which occurs when the prevailing winds are along the coast from the north-west. Figure 2 may also be compared with figure 302 of Defant (1961) which shows a cross-section of the Benguela current off south-west Africa. The direction of the currents are reversed due to the change of hemisphere. Clearly more realistic models must take account of the temperature changes brought about by the upwelling.

6. The west coast boundary layer

To complete the ocean circulation a boundary layer is also required along the west coast. This boundary layer has two purposes, first, to provide a northwards flow when there is a southward flow in the interior and secondly to provide upwelling when there is downwelling at the east coast.

In the outer $E^{\frac{1}{3}}$ layer, the stretched co-ordinate is now defined as $\eta = xE^{-\frac{1}{3}}$ and the equation corresponding to (13) is

$$\hat{v}_{\eta\eta\eta} - \beta\hat{v} = 0.$$

The change in sign alters the nature of the solution, which is given by Pedlosky (1968), and is the standard westward intensification solution with an intense northwards current of magnitude $O(E^{\frac{1}{3}})$.

In the inner $(\delta E)^{\frac{1}{3}}$ layer, the appropriate co-ordinate is $\xi = x(\delta E)^{-\frac{1}{3}}$ and the equation of motion is

$$\left(\frac{\partial^6}{\partial \xi^6}+f^2\frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial \overline{u}_0}{\partial z}, v_0, \overline{w}_0, \overline{P}_1\right) = 0,$$

unchanged from (22). Thus the nature of the solution is the same as in the eastern boundary layer. In particular the vertical velocity is

$$w = -\frac{E^{\frac{1}{6}}}{\delta^{\frac{1}{3}}}\tau^y(0,y)\sum_{n=1}^{\infty}(4/\sqrt{3})\gamma_n^{-2}\sin(n\pi z)\exp(-\tfrac{1}{2}\gamma_n\xi)\sin\tfrac{1}{2}\sqrt{3}\,\gamma_n\xi,$$

very similar to (29). This is essentially an upwelling layer in which water from the interior is returned to the Ekman layer. There is an associated north–south secondary flow of magnitude $O(\delta^{\frac{2}{3}}E^{\frac{1}{6}})$ but this is small compared with the $O(E^{\frac{1}{6}})$ flow in the thicker layer.

7. An ocean circulation with zonal wind stress

In order to examine the implications of the above result to a full ocean model, consider the application of an x independent zonal wind stress. Then the Ekman-layer flux is independent of x and

$$u_I = E^{\frac{1}{2}}\left\{\frac{(x-1)}{\beta}\frac{\partial^2\tau^x}{\partial y^2}-\frac{\tau^y}{f}\right\} \tag{30}$$

from (8) and (26). An interesting feature arises in the interior from inclusion in (30) of fluid that comes from downwelling at the east coast. We see from (30) that

$$u_I = 0 \quad \text{when} \quad x = 1+\frac{\beta}{f}\tau^y\bigg/\frac{\partial^2\tau^x}{\partial y^2} = 1-x_0(y). \tag{31}$$

The implications of (31) may be illustrated by using a particular wind stress distribution. Consider the application of the wind stress

$$\tau^x = -\cos(\pi y/b)\cos\theta, \quad \tau^y = -\cos(\pi y/b)\sin\theta \tag{32}$$

to the rectangular ocean described in §1. This represents a zonal wind stress that is inclined at the angle θ to the north of east and is used to represent the trade winds and westerlies over the region $0 \leqslant y \leqslant 1$. From (8) to (10) and (26) this wind stress gives rise to the following interior flow

$$u_I = E^{\frac{1}{2}}\cos\frac{\pi y}{b}\left\{\frac{1}{f}\sin\theta+\frac{\pi^2}{\beta b^2}(x-1)\cos\theta\right\}, \tag{33}$$

$$v_I = -E^{\frac{1}{2}}\left\{\frac{\pi}{\beta b}\sin\pi\frac{y}{b}+\frac{1}{f}\cos\frac{\pi y}{b}\right\}\cos\theta,$$

$$w_I = (z+1)\frac{\beta}{f}v_I.$$

We have from (33) that

$$u_I = 0 \quad \text{when} \quad x = 1-(b^2\beta/\pi^2 f)\tan\theta,$$

and $x_0(y)$ is small if $\tan\theta$ is less than $\pi^2 f/b^2\beta$. Now for a typical ocean gyre the north–south length scale is not larger than the east–west length scale and so $b \leqslant 1$. Hence if $f/b^2\beta \sim O(1)$ then x_0 is small if θ is less than 84°.

In the principal gyre to the north of $y = 0$, v_I is southwards. Hence on the curve $x = 1 - x_0$ where $u_I = 0$ the flow is southwards into a secondary gyre near the east coast, as shown in figure 3 for $\theta = \frac{1}{4}\pi$, $b = 1$. Thus for this wind stress distribution *water that downwells at the east coast penetrates only a short way into the interior. The oceanic circulation between two latitudes of zero wind stress curl splits into two gyres, a secondary narrow gyre being formed near the east coast.* Also if $\tan\theta$ is less than $\pi^2 f/b^2\beta$, (33) shows that the cosine term will dominate the sine term at $x = 0$ and then water that downwells at the west coast can never enter the interior but merely reinforces the north–south transport in the $E^{\frac{1}{3}}$ layer. These results for the interior flow hold for all values of the ratio E_H/δ^2.

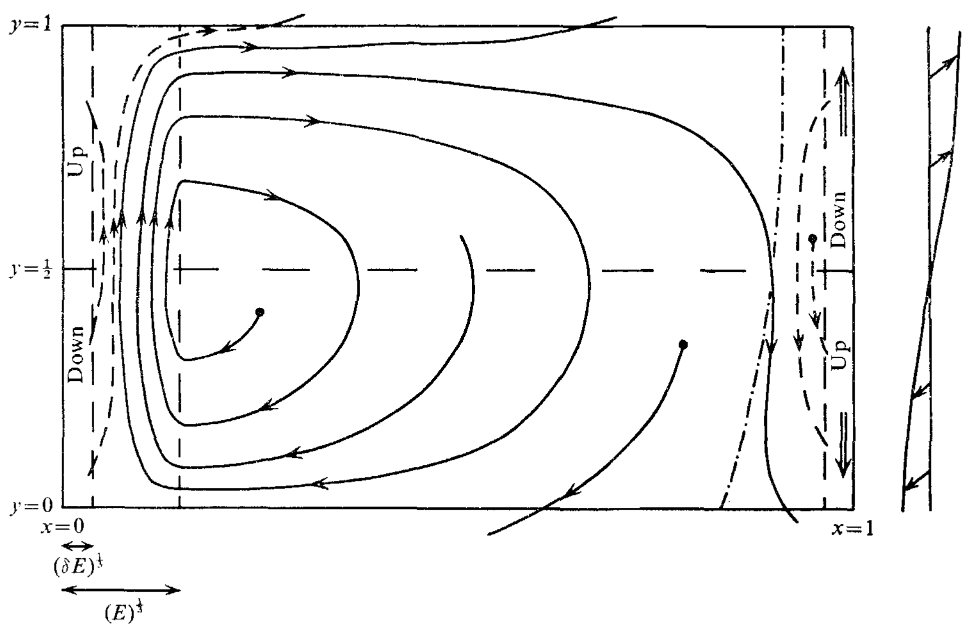

FIGURE 3. Diagrammatic representation of the horizontal flow below the Ekman layer. There is downflow from the Ekman layer into the interior region. The open arrows indicate the direction of the surface east coast current. Upwelling and downwelling are indicated in the $(\delta E)^{\frac{1}{3}}$ layer. The wind stress distribution is shown as the right. The curve $x = 1 - x_0(y)$ is shown by —·—·—.

In § 4 we show that the east coast current is northwards near the surface when $\tau^y(1, y)$ is positive. Thus for this wind stress the current near the surface is southwards for $0 \leqslant y < \frac{1}{2}$ and northwards for $\frac{1}{2} < y \leqslant 1$. The direction is reversed at lower depths.

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Appendix. Derivation of the solution for the eastern boundary layer

To find the solution of (22), let $\overline{w}_0 = g(z)\,h(y,\xi)$, and then (22) gives

$$g(\partial^6 h/\partial\xi^6) + f^2 g'' h = 0,$$

which separates into

$$g'' + A^6 g = 0, \quad \partial^6 h/\partial\xi^6 - A^6 f^2 h = 0,$$

where A is a constant. The bounded solution of these equations which satisfies (25) as $\xi \to \infty$ is

$$\overline{w}_0 = \sum_{A\neq 0} (\overline{w}_{01}\cos A^3 z + \overline{w}_{02}\sin A^3 z)\exp(-\tfrac{1}{2}Af^{\frac{1}{3}}\xi) \times\{\overline{w}_{03}\exp(-\tfrac{1}{2}Af^{\frac{1}{3}}\xi) + \overline{w}_{04}\cos\tfrac{1}{2}\sqrt{3}\,Af^{\frac{1}{3}}\xi + \overline{w}_{05}\sin\tfrac{1}{2}\sqrt{3}\,Af^{\frac{1}{3}}\xi\}. \quad \text{(A 1)}$$

The sum is taken over all permissible values of A, the zero value being excluded by (25). The conditions (23) require that

$$\overline{w}_{01} = 0 \quad \text{and} \quad A^3 = n\pi \quad (n = 1, 2, \ldots)$$

as the terms in ξ are linearly independent for the different values of A, and (24) requires that

$$\overline{w}_{03} + \overline{w}_{04} = 0.$$

Hence, choosing $\overline{w}_{02} = 1$, the solution for $\overline{w}_0$ reduces to

$$\overline{w}_0 = \sum_{n=1}^{\infty} \sin n\pi z\, e^{-\frac{1}{2}\gamma_n\xi}\{\overline{w}_{03}e^{-\frac{1}{2}\gamma_n\xi} - \overline{w}_{03}\cos\tfrac{1}{2}\sqrt{3}\gamma_n\xi + \overline{w}_{05}\sin\tfrac{1}{2}\sqrt{3}\gamma_n\xi\}, \quad \text{(A 2)}$$

where $\gamma_n = (n\pi f)^{\frac{1}{3}}$.

The solutions of (22) for $\partial\overline{u}_0/\partial z$ and $\hat{v}_0$ may be found in the same form as (A 1) and as they must satisfy (18), (20), (21) involving $\overline{w}_0$, we see that $A^3 = n\pi$ again. The solution for $\overline{v}_0$ satisfying (24) and (25) is

$$\overline{v}_0 = \sum_{n=1}^{\infty} (\overline{v}_{01}\cos n\pi z + \overline{v}_{02}\sin n\pi z)\, e^{-\frac{1}{2}\gamma_n\xi} \times\{\overline{v}_{03}e^{-\frac{1}{2}\gamma_n\xi} - \overline{v}_{03}\cos\tfrac{1}{2}\sqrt{3}\gamma_n\xi - \overline{v}_{05}\sin\tfrac{1}{2}\sqrt{3}\gamma_n\xi\}.$$

The relationship between $\overline{v}_0$ and $\overline{w}_0$ is given from (18) and (20) as

$$-f\overline{v}_{0z} = \overline{w}_{0\xi\xi\xi}. \quad \text{(A 3)}$$

As these series may not be uniformly convergent near $z = 0$, $\xi = 0$ where the Ekman-layer transport enters this boundary layer we may not differentiate these series but it is permissible to integrate them.

Therefore multiplication of (A 3) by $\sin m\pi z$ and integration by parts leads to

$$-[f\overline{v}_0\sin m\pi z]_{-1}^{0} + \int_{-1}^{0} m\pi f\overline{v}_0\cos m\pi z\,dz = \frac{\partial^3}{\partial\xi^3}\int_{-1}^{0}\overline{w}_0\sin m\pi z\,dz.$$

Substituting for $\overline{v}_0$ and $\overline{w}_0$, using the orthogonality relations, and equating coefficients, we have

$$\overline{v}_{02} = 0, \quad \overline{v}_{01}\overline{v}_{03} = -\overline{w}_{03}, \quad \overline{v}_{01}\overline{v}_{03} = \overline{w}_{03}, \quad \overline{v}_{01}\overline{v}_{05} = \overline{w}_{05}.$$

Hence
$$\overline{v}_0 = \sum_{n=1}^{\infty} \overline{w}_{05}\cos n\pi z\, e^{-\frac{1}{2}\gamma_n\xi}\sin\tfrac{1}{2}\sqrt{3}\,\gamma_n\xi, \quad \text{(A 4)}$$

and
$$\overline{w}_0 = \sum_{n=1}^{\infty} \overline{w}_{05}\sin n\pi z\, e^{-\frac{1}{2}\gamma_n\xi}\sin\tfrac{1}{2}\sqrt{3}\gamma_n\xi. \quad \text{(A 5)}$$

Equation (21) gives the relationship

$$\bar{u}_{0\xi z} = \bar{w}_{0zz},$$

which may be integrated as before to give

$$\frac{\partial \bar{u}_0}{\partial z} = \sum_{n=1}^{\infty} \frac{(n\pi)^{\frac{5}{3}}}{f^{\frac{1}{3}}} \bar{w}_{05} \sin n\pi z\, e^{-\frac{1}{2}\gamma_n \xi} \sin(\tfrac{1}{2}\sqrt{3}\,\gamma_n \xi + \tfrac{1}{3}\pi).$$

The integral of this is

$$\bar{u}_0 = U(y) - \sum_{n=1}^{\infty} \frac{(n\pi)^{\frac{2}{3}}}{f^{\frac{1}{3}}} \bar{w}_{05} \cos n\pi z\, e^{-\frac{1}{2}\gamma_n \xi} \sin(\tfrac{1}{2}\sqrt{3}\,\gamma_n \xi + \tfrac{1}{3}\pi), \tag{A 6}$$

where use has been made of the matching condition (25).

To find $\bar{w}_{05}$ we use the condition on the volume flux through the corner region given by (26). Now (21) may be integrated to

$$\bar{u}_0 = \int_0^{\xi} \bar{w}_{0z}\, d\xi',$$

which if multiplied by $\cos m\pi z$ and integrated with respect to z, gives

$$\int_{-1}^{0} \bar{u}_0 \cos m\pi z\, dz = \int_{-1}^{0} \int_0^{\xi} \bar{w}_{0z}\, d\xi' \cos m\pi z\, dz$$
$$= \int_0^{\xi} [\bar{w}_0(\xi', y, 0) - (-1)^m w_0(\xi', y, -1)]\, d\xi + \int_{-1}^{0} \int_0^{\xi} \bar{w}_0\, d\xi'\, m\pi \sin m\pi z\, dz.$$

Therefore, using (A 5) and (A 6),

$$-\frac{(m\pi)^{\frac{2}{3}}}{2f^{\frac{1}{3}}} \bar{w}_{05}\, e^{-\frac{1}{2}\gamma_m \xi} \sin(\tfrac{1}{2}\sqrt{3}\gamma_m \xi + \tfrac{1}{3}\pi)$$
$$= \int_0^{\xi} \bar{w}_0(\xi', y, 0)\, d\xi' + \int_0^{\xi} \tfrac{1}{2} m\pi \bar{w}_{05}\, e^{-\frac{1}{2}\gamma_m \xi'} \sin \tfrac{1}{2}\sqrt{3}\,\gamma_m \xi'\, d\xi'$$
$$= \int_0^{\xi} \bar{w}_0(\xi', y, 0)\, d\xi' - \left[\tfrac{1}{2} m\pi \frac{\bar{w}_{05}}{\gamma_m^{\frac{1}{3}}} e^{-\frac{1}{2}\gamma_m \xi'} \sin \tfrac{1}{2}\sqrt{3}\,\gamma_m \xi + \tfrac{1}{3}\pi) \right]_0^{\xi},$$

where $\gamma_m = (m\pi f)^{\frac{1}{3}}$. Now let $\xi \to \infty$ and use (26), giving in the limit

$$\bar{w}_{05} = \frac{4}{\sqrt{3}} \frac{\tau^y(1, y)}{\gamma_m^{\frac{2}{3}}}. \tag{A 7}$$

Substitution of (A 7) into (A 6), (A 4) and (A 5) leads to (27), (28), and (29) respectively.

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